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Conditional matching preclusion for the arrangement graphs

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ABSTRACT

The matching preclusion number of a graph is the minimum number of edges whose deletion results in a graph that has neither perfect matchings nor almost-perfect matchings. For many interconnection networks, the optimal sets are precisely those induced by a single vertex. Recently, the conditional matching preclusion number of a graph was introduced to look for obstruction sets beyond those induced by a single vertex. It is defined to be the minimum number of edges whose deletion results in a graph *with no isolated vertices* that has neither perfect matchings nor almost-perfect matchings. In this paper we find this number and classify all optimal sets for the arrangement graphs, one of the most popular interconnection networks.

1. Introduction and Preliminaries

A set of edges in a graph is called a *matching* if every vertex is incident to at most one edge in this set (the edges of such a set are called *independent*). A *perfect matching* is a set of edges such that every vertex is incident to exactly one edge in this set. An *almost-perfect matching* is a set of edges such that every vertex except one is incident to exactly one edge in this set and the exceptional vertex is incident to none. So if a graph has a perfect matching, then it has an even number of vertices; if a graph has an almost-perfect matching, then it has an odd number of vertices. The *matching preclusion number* of a graph G, denoted by $\mathrm{mp}(G)$, is the minimum number of edges whose deletion leaves the resulting graph without a perfect matching or an almost-perfect matching. Any such optimal set is called an *optimal matching preclusion set*. So $\mathrm{mp}(G) = 0$ if G has neither a perfect matching nor an almost-perfect matching. This concept of matching preclusion can be traced back to a result of Plesník [28] although it was formally introduced and defined by Brigham et al. [3], and it was further studied in [9,6,25,31]. Matching theory is a major topic in discrete mathematics and has important applications in theoretical computer science; [4,24] provide a small sample of research in this area. The concept of matching preclusion was introduced as a measure of robustness in the event of edge failure in interconnection networks, as well as a theoretical connection to conditional connectivity, "changing and unchanging of invariants" and extremal graph theory. We refer the readers to [3] for details and additional references.

Useful distributed processor architectures offer the advantage of improved connectivity and reliability. An important component of such a distributed system is the system topology, which defines the inter-processor communication architecture which is an interconnection network. In certain applications, when one wants to limit the communications between far apart processors, the interconnection network may be divided into clusters and most communications are within a cluster. As a special case, every processor requires a special partner at any given time (when most communications

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occur between a processor and its special partner) and the matching preclusion number measures the robustness of this requirement in the event of link failures as indicated in [3]. (It is also possible to consider vertex failure, see [27].) Hence in these interconnection networks, it is desirable to have the property that the only optimal matching preclusion sets are those whose elements are incident to a single vertex.

Proposition 1.1. *Let G be a graph with an even number of vertices. Then $mp(G) \leq \delta(G)$, where $\delta(G)$ is the minimum degree of G.*

Proof. Deleting all edges incident to a single vertex will give a graph with no perfect matchings and the result follows. □

We call an optimal solution of the form given in the proof of Proposition 1.1 a *trivial optimal matching preclusion set*. As mentioned earlier, it is desirable for an interconnection network to have only trivial optimal matching preclusion sets. Given that it is unlikely that in the event of random link failure, all of them will be at the same vertex, it is natural to ask what are the next obstruction sets for a graph with link failures to have a perfect matching subject to the condition that the *faulty graph* has no isolated vertices. This motivates the following definition given in [7]: let G be a graph in which $\mathrm{mp}(G) > 0$. The *conditional matching preclusion number* of G, denoted by $\mathrm{mp}_1(G)$, is the minimum number of edges whose deletion leaves the resulting graph with no isolated vertices and without a perfect matching or almost-perfect matching. Any such optimal set is called an *optimal conditional matching preclusion set*. If a conditional matching preclusion set does not exist in G, that is, we cannot delete edges to satisfy both conditions, we leave $\mathrm{mp}_1(G)$ undefined. In this paper the graphs that we consider have conditional matching preclusion sets.

So the question is: if we delete edges, what are the basic obstructions to a perfect matching or an almost-perfect matching in the resulting graph if no isolated vertices are created? Proposition 1.1 shows that without the condition of no isolated vertices, an isolated vertex will be a basic obstruction, and deleting all edges incident to G will produce a trivial matching preclusion set. For a graph with no isolated vertices, a basic obstruction to a perfect matching will be the existence of a path u–w–v, where the degree of u and the degree of v are both 1. So to produce such an obstruction set, one can pick any path u–w–v in the original graph and delete all the edges incident to either u or v but not w. We define

$$\nu_e(G) = \min\{d_G(u) + d_G(v) - 2 - y_G(u, v) : \text{ there is a 2-path between } u \text{ and } v\},$$

where $d_G(\cdot)$ is the degree function, and $y_G(u, v) = 1$ if u and v are adjacent and 0 otherwise. (We will suppress G and simply write d and y if it is clear from the context.) So mirroring Proposition 1.1, we have the following result:

Proposition 1.2. *Let G be a graph with an even number of vertices. Suppose that every vertex in G has degree at least 3. Then $mp_1(G) \leq \nu_e(G)$.*

We note that the condition $\delta(G) \geq 3$ in Proposition 1.2 is to ensure that the resulting graph (after edges have been deleted) has no isolated vertices. (Although the condition can be weakened, it suffices for our purpose.) We call an optimal solution of the form induced by ν_e a *trivial optimal conditional matching preclusion set*. As mentioned earlier, the matching preclusion number measures the robustness of this requirement in the event of link failures, so it is desirable to have the property that all optimal matching preclusion sets are trivial. Similarly, it is desirable to have the property that all optimal conditional matching preclusion sets are trivial as well. [7] introduced this concept and considered the conditional matching preclusion problem for a number of basic networks including the hypercubes, and it was proved that they have this desired property. This problem was also studied in [26] for hypercube-like interconnection networks and in [31] for k-ary n-cubes. In this paper we investigate this property for the arrangement graphs. Suppose $n \geq 3$ and $2 \leq k \leq n-1$. The vertex set of the arrangement graph $A_{n,k}$ is the set of k-permutations from $\{1, 2, \ldots, n\}$, and two vertices $[a_1, a_2, a_3, \ldots, a_k]$ and $[b_1, b_2, b_3, \ldots, b_k]$ are adjacent if they differ in exactly one position. Although the hypercubes form a popular class of interconnection networks, there are other classes with better properties. The class of star graphs [1] and the class of alternating group graphs [21] were introduced to outperform the hypercubes. The class of arrangement graphs $A_{n,k}$ was introduced in [13] to be a common generalization of the star graphs and the alternating group graphs, and to provide an even richer class of interconnection networks. We refer the readers to [13] for additional benefits. Indeed, the graph $A_{n,n-1}$ is isomorphic to the star graph S_n given in [1] and $A_{n,n-2}$ is isomorphic to the alternating group graph A_n given in [21]. It is well-known, and indeed easy to see, that $A_{n,k}$ is a vertex transitive, $k(n-k)$-regular graph with $n!/(n-k)!$ vertices. Fig. 1 shows $A_{4,2}$. (For convenience, we write the $(n, 2)$-permutation $[i, j]$ as ij in this figure, for example $[1, 4]$ as 14.) Like the hypercubes, there have been a lot of research on this class and related classes of interconnection networks including embeddings, Hamiltonicity and surface area as well as their applicability in theoretical computer science [12,2,14–16,18,30,23,10,11,22,17,19,5,29].

A standard way to view $A_{n,k}$ is via its recursive structure. Let H_i be the subgraph of $A_{n,k}$ induced by the vertices whose last symbol is i. Then H_i is isomorphic to $A_{n-1,k-1}$. Every vertex v in H_i has exactly $n-k$ neighbors outside of H_i; moreover, its $n-k$ neighbors belong to different H_j's. To be precise, if $v = [a_1, a_2, \ldots, a_k]$, then its $n-k$ neighbors not in H_{a_k} belong to different H_j's, where $j \in \{1, 2, \ldots, n\} - \{a_1, a_2, \ldots, a_k\}$. We call these neighbors the *outside neighbors* of v. We call the edges whose end-vertices belong to different H_j's *cross edges*. Since the H_i's are defined via the kth position, we say that it is a *decomposition via the kth position*. It is easy to see that for a given pair of H_i and H_j, there are $(n-2)!/(n-k-1)!$ cross edges between them; moreover, they are independent. There is nothing special about the last position, we can instead decompose via any one of the k positions. This standard recursive decomposition is a useful way to solve problems arising from the arrangement graphs.

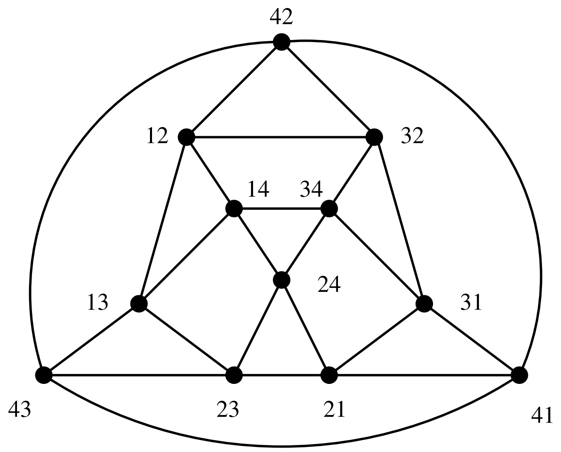

Fig. 1. $A_{4,2}$.

We need some known results including those for the matching preclusion problem, special cases and a Hamiltonicity result.

Theorem 1.3 ([6,9]). *Suppose $n \geq 4$. Then $mp(A_{n,k}) = k(n-k)$. Moreover, every optimal matching preclusion set is trivial.*

Theorem 1.4 ([8]). *Suppose $n \geq 4$. Then $mp_1(A_{n,n-2}) = 4n - 11$. Moreover, every optimal conditional matching preclusion set is trivial.*

Theorem 1.5 ([20]). *Let $n \geq 4$ and $2 \leq k \leq n-2$. Suppose $F \subseteq V(A_{n,k}) \cup E(A_{n,k})$. If $|F| \leq k(n-k) - 3$, then $A_{n,k} - F$ is Hamiltonian connected.*[1] *If $|F| \leq k(n-k) - 2$, then $A_{n,k} - F$ is Hamiltonian.*

We do not need the full strength of Theorem 1.5. We will record the special cases (mostly by deleting edges only) that we need as a corollary.

Corollary 1.6. *Let $n \geq 4$, $2 \leq k \leq n-2$ and $(n, k) \neq (4, 2)$. If $F \subseteq E(A_{n,k})$ such that $|F| \leq k(n-k) - 3$, then $A_{n,k} - F$ is Hamiltonian connected; if $|F| \leq k(n-k) - 2$, then $A_{n,k} - F$ is Hamiltonian. If z_1, z_2, z_3 are three mutually adjacent vertices in $A_{n,k}$ and f is an edge in $A_{n,k}$, then $A_{n,k} - \{z_1, z_2, z_3\}$ and $A_{n,k} - \{z_1, z_2, z_3, f\}$ are Hamiltonian.*

Proof. The first statement follows directly from Theorem 1.5. The second statement is also a direct consequence as $k(n-k) - 2 \geq 4$ for $k \geq 2$, $n-k \geq 2$ and $(n, k) \neq (4, 2)$. $\square$

Since the alternating group graph A_n is isomorphic to $A_{n,n-2}$, one may wonder what the differences are between the proof for A_n given in [8] and the proof for $A_{n,k}$ given here. The proof given here is not a straightforward generalization of the argument given in [8] even though both proofs use induction. For A_n, one only has to check a couple of graphs to establish the base case. Here the base case is a class with infinitely many graphs, that is, $A_{n,2}$ for $n \geq 4$. In the induction step where we progress from $A_{n-1,k-1}$ to $A_{n,k}$, the argument requires substantial modification of the proof for $A_{n,n-2}$. Indeed, the argument here requires the assumption that $n - k \geq 3$. This is not surprising as with the two parameters, the boundaries are $A_{n,2}$ and $A_{n,n-2}$, and it is typical that one needs a dedicated argument for each of the boundaries as well as a separate argument for the interior. So we leverage Theorem 1.4 from [8] for one boundary case. In Section 3, we present the other boundary case, and the main induction proof is given in Section 2.

2. The main result

We now present our main result. It may seem out of order for us to present the proof of this result before proving the remaining boundary case. However, the proof of the boundary case is actually much more involved, so we choose to invert the order of presentation.

Theorem 2.1. *Let $n \geq 4$ and $2 \leq k \leq n-2$. Then $mp_1(A_{n,k}) = 2k(n-k) - 3$. Moreover, every optimal conditional matching preclusion set is trivial.*

Proof. It is enough to prove that if at most $2k(n-k) - 3$ edges are deleted from $A_{n,k}$, then the resulting graph must satisfy one of following: (1) it has a perfect matching, (2) it has an isolated vertex, or (3) the deleted edges form a trivial conditional matching preclusion set. If $n = 4$, then $k = 2$, so this case has been covered by Theorem 1.4. If $n = 5$, then the case $A_{5,2}$ will be covered by Theorem 3.1 and the case $A_{5,3}$ is covered by Theorem 1.4. Moreover, the case $A_{n,2}$ will be covered by Theorem 3.1. We proceed with induction on n. Assume $n \geq 6$ and $k \geq 3$. We may also assume that $n - k \geq 3$ as the case

[1] A graph is Hamiltonian connected if there is a Hamiltonian path between every pair of vertices.

$n-k=2$ is covered by Theorem 1.4. Let F be a set of edges of size at most $2k(n-k)-3$; these edges will be called *faulty*. If (1) or (2) is true, then we are done. So we may assume that $A_{n,k}-F$ has no isolated vertices and has no perfect matchings. It is easy to see that we may assume $|F|=2k(n-k)-3$, since otherwise we can easily delete additional edges without violating (2) or (3), and $A_{n,k}-F$ will still have no perfect matchings. We want to prove that F is a trivial conditional matching preclusion set.

We will choose a position to decompose $A_{n,k}$ such that the number of faulty cross edges is maximized. Since we can decompose it along k positions, one of them has at least $\lceil\frac{2k(n-k)-3}{k}\rceil = 2(n-k)-\lfloor\frac{3}{k}\rfloor$ cross edges. If $k\geq 4$, then this number is $2(n-k)$. If $k=3$, then this number is $2(n-k)-1$. For notational convenience, we may assume that the decomposition is through the last position. Let H_i be the subgraph of $A_{n,k}$ with i in the last position for $1\leq i\leq n$. So H_i is isomorphic to $A_{n-1,k-1}$. Let B be the set of cross edges. Then $|B\cap F|\geq 2(n-k)-\delta$ where $\delta=0$ if $k\geq 4$ and $\delta=1$ if $k=3$. So there are at most $2k(n-k)-3-2(n-k)+\delta=2(k-1)(n-k)-3+\delta$ faults in the H_i's. Now, at least one of the H_i-F's has no perfect matchings, otherwise the union of these matchings is a perfect matching of $A_{n,k}-F$. If H_i-F has no perfect matchings, it must contain at least $(k-1)(n-k)$ faulty edges by Theorem 1.3. Hence there is exactly one such H_i as $2(k-1)(n-k)>2(k-1)(n-k)-3+\delta$. For notational convenience, we assume it is H_1. Note also that H_1 contains at most $2(k-1)(n-k)-3+\delta$ faulty edges. Moreover, each of the other H_i's contains at most $2k(n-k)-3-(k-1)(n-k)-2(n-k)+\delta=(k-1)(n-k)-3+\delta<(k-1)(n-k)$ faulty edges, and hence H_i-F has a perfect matching for $i\neq 1$ by Theorem 1.3. We consider two cases.

Case 1: H_1-F has an isolated vertex.

Let u be such an isolated vertex. We note that this is unique as H_1-F cannot have two isolated vertices; otherwise, we need to delete at least $2(k-1)(n-k)-1$ edges in H_1, but $2(k-1)(n-k)-1>2(k-1)(n-k)-3+\delta$. There are $n-k$ cross edges incident to u, and at least one of them is not a fault as $A_{n,k}-F$ has no isolated vertices. Let (u,u') be such a cross edge. Again, for notational convenience, we assume that u' is in H_2. There are $(n-2)!/(n-k-1)!$ independent cross edges between H_1 and H_2. The number of cross edges in F is at most $2k(n-k)-3-(k-1)(n-k)=(k+1)(n-k)-3$. We claim that we can find a cross edge between H_1 and H_2 that is neither a fault nor (u,u'). The claim can be established by showing $(n-2)!/(n-k-1)!>(k+1)(n-k)-2$ or equivalently $(n-2)!>(k+1)(n-k)!-2(n-k-1)!$, which is true since $n-k-1>0$, $(n-3)!\geq(n-k)!$ as $k\geq 3$, and $n-2\geq k+1$ as $n-k\geq 3$. Let (v,v') be such a cross edge with v in H_1 and v' in H_2. Let $F_1=F\cap E(H_1)$ and $F_2=F\cap E(H_2)$. Then $(k-1)(n-k)\leq|F_1|\leq 2(k-1)(n-k)-3+\delta$ and $|F_2|\leq(k-1)(n-k)-3+\delta$. Let F_1' be obtained from F_1 by deleting the edges in H_1 that are incident to u (that is, they are no longer faults). Then $|F_1'|\leq 2(k-1)(n-k)-3+\delta-(k-1)(n-k)=(k-1)(n-k)-3+\delta$.

Suppose $\delta=0$. Then by Corollary 1.6, there is a Hamiltonian path P_1 in H_1 from u to v and a Hamiltonian path P_2 in H_2 from u' to v'. Now, P_1 contains at most one element of F_1-F_1'. Since P_1, P_2, (v,v'), (u,u') form an even cycle spanning all vertices of H_1 and H_2, it contains a fault-free perfect matching of the vertices of H_1 and H_2. Now for $i\geq 3$, H_i-F has a perfect matching as observed earlier, hence $A_{n,k}-F$ has a perfect matching, a contradiction.

We now consider the case $\delta=1$. Then $k=3$. The above argument is still valid unless $|B\cap F|=2(n-k)-1=2(n-3)-1=2n-7$, and either $|F_1'|=(k-1)(n-k)-3+1=2(n-3)-2=2n-8$ or $|F_2|=(k-1)(n-k)-3+1=2(n-3)-2=2n-8$. Note that $|F|=2k(n-k)-3=6(n-3)-3=6n-21$; moreover, $2n-6$ of these edges are incident to u in H_1 and exactly $2n-7$ of them are cross edges. So there are only $2n-8$ faulty edges left, hence either $|F_1'|=2n-8$ and $|F_2|=0$, or $|F_1'|=0$ and $|F_2|=2n-8$. Since the argument that we now give is symmetric with respect to F_1' and F_2, we assume that $|F_1'|=2n-8$. Note that each H_i with $i\neq 1$ has no faults. The above argument does not apply as v in H_1 is preselected, but we cannot guarantee a Hamiltonian path between u and v in H_1-F_1'. However, we can relax the requirement as almost any Hamiltonian path starting at u will do with the caveat that we have to look at two subcases. Since H_1 is isomorphic to $A_{n-1,2}$ and $|F_1'|=2(n-3)-2$, there is a Hamiltonian cycle in H_1-F_1' by Corollary 1.6. Consider the two neighbors of u on C, each of which is incident to $n-3$ cross edges for a total of $2n-6$ cross edges. But there are only $2n-7$ cross edges that are faulty, so one of them is not in F. Let y be a neighbor of u on C and let (y,y') be a cross edge that is not in F. Suppose that y' is in H_2. Then as before, we first obtain a Hamiltonian path P_2 between v' and y'. Now $C-\{u,y\}$ induces a perfect matching in $H_1-(\{u,y\}\cup F_1)$, and $P_2-\{u',y'\}$ induces a perfect matching in $H_2-\{u',y'\}$. This together with (u,u'), (y,y') and a perfect matching from each H_i for $i\neq 1,2$ gives a perfect matching in $A_{n,3}-F$, a contradiction. Suppose y' is in another H_i, say H_3, for notational convenience. Then using the usual argument, we can get a cross edge (z,z') between H_2 and H_3 such that it is not faulty, z is in H_2, z' is in H_3, $z\neq u'$ and $z'\neq y'$. Now we obtain a Hamiltonian path P_2 between u' and z and a Hamiltonian path P_3 between y' and z'. So $C-\{u,y\}$ induces a perfect matching in $H_1-(\{u,y\}\cup F_1)$, $P_2-\{u',z\}$ induces a perfect matching in $H_2-\{u',z\}$, and $P_3-\{y',z'\}$ induces a perfect matching in $H_3-\{y',z'\}$ This together with (u,u'), (y,y'), (z,z') and a perfect matching from each H_i for $i\neq 1,2,3$ gives a perfect matching in $A_{n,3}-F$, a contradiction. (We note that the argument for $k=3$ can be extended to the earlier case as well. However, we chose to separate it into the two arguments to illustrate that the case $k\geq 4$ is easier.)

Case 2: H_1-F has no isolated vertices.

Since $\mathrm{mp}_1(H_1)=2(k-1)(n-k)-3$, H_1 contains at least $2(k-1)(n-k)-3$ faulty edges by the induction hypothesis as H_1-F has no perfect matchings. So we have found at least $2(k-1)(n-k)-3+2(n-k)-\delta=2k(n-k)-3-\delta$ faulty edges. If $k\geq 4$, then H_1 has exactly $2(k-1)(n-k)-3$ faulty edges, exactly $2(n-k)$ cross edges are faulty, and $H_2,H_3,\ldots,H_n$ have no faulty edges. Then by the induction hypothesis, the $2(k-1)(n-k)-3$ faulty edges in H_1 form a trivial conditional matching preclusion set in H_1. So in H_1 there is a path u–w–v with u and v adjacent such that the $2(k-1)(n-k)-3$ faulty edges in H_1 are precisely the edges incident to either u or v not including (u,w) and (v,w) (they do include (u,v)). Each of u and v

are incident to $n-k$ cross edges. If these $2(n-k)$ cross edges are precisely the $2(n-k)$ cross edges that are faulty, then F is a trivial conditional matching preclusion set. So we may assume that this is not the case, so a cross edge (u, u') is not faulty. For notational convenience, assume that u' is in H_2. Now, by Corollary 1.6, $H_1 - \{u, w, v\}$ has a Hamiltonian cycle C. Note that C is an odd cycle. Clearly, we can find a vertex y on C such that (y, y') is a cross edge that is not a fault and that y' is in H_2. (There are $(n-2)!/(n-k-1)!$ independent edges between H_1 and H_2. At most $2(n-k)$ of them are fault edges and at most three of them are incident to one of u, v, w. Now the claim follows if we can check that $(n-2)!/(n-k-1)! > 2(n-k)+3$ or equivalently $(n-2)! > 2(n-k)! + 3(n-k-1)!$. To see this, write $(n-2)! = 2(n-3)! + (n-4)(n-3)!$. Now $2(n-3)! \geq 2(n-k)!$ and $(n-4)(n-3)! \geq 2(n-k)! > 3(n-k-1)!$.) Let P be a Hamiltonian path from u' to y' in H_2. Now it is easy to get a perfect matching for $A_{n,k}$: We use (w, v), (y, y'),(u, u'), a perfect matching induced by $C - \{y\}$, a perfect matching induced by $P - \{u', y'\}$, and a perfect matching from each H_i for $i = 3, 4, \ldots, n$.

We now consider the case $k = 3$. Then H_1 can have either $2(k-1)(n-k)-3 = 4(n-3)-3$ or $4(n-3)-2$ faulty edges. If it has $4(n-3)-3$ faulty edges, then we can simply repeat the above argument with minimal changes. In particular, we have either exactly $2(n-k) = 2(n-3)$ cross edges that are faulty and $H_2, H_3, \ldots, H_n$ have no faulty edges, or we have exactly $2(n-3)-1$ cross edges that are faulty and $H_2, H_3, \ldots, H_n$ have exactly one faulty edge. In both cases the above argument applies, since having one faulty edge in $H_2, H_3, \ldots, H_n$ still allows us to find the necessary Hamiltonian path. Now consider the case when H_1 has $4(n-3)-2$ faulty edges. Then we have exactly $2(n-3)-1$ faulty cross edges, and $H_2, H_3, \ldots, H_n$ have no faulty edges. Again we modify our previous argument. Let F_1 be the faulty edges in H_1 and pick $f \in F_1$. Let $F_1' = F_1 - \{f\}$. We claim that we can pick f such that $H_1 - F_1'$ has no isolated vertices and no perfect matchings. Clearly, for any choice of f, the resulting $H_1 - F_1'$ will have no isolated vertices. Thus we need only find an $f \in F_1$ whose re-addition to the graph produces no perfect matching in $H_1 - F_1'$. Each edge in F_1 shares an endpoint with exactly $2n-6$ cross edges. Call $f \in F_1$ a *candidate* if both of its endpoints have cross edges which are not in F. Then f can only fail to be a candidate if all $n-3$ cross edges belonging to one of its endpoints are faulty edges. As the number of cross edges is $2(n-3)-1$, this can clearly only be true for at most one vertex in H_1, which is incident to $2n-6$ edges in H_1. Since $|F_1| = 4(n-3)-2 > 2n-6$, there must be a candidate edge. Thus let f be such a candidate edge, which we claim will suffice. Assume, by way of contradiction, that $H_1 - F_1'$ has a perfect matching. Then obviously f is in this matching, because $H_1 - F_1$ does not have a perfect matching. Let f have endpoints u, v with cross edges (u, u') and (v, v') not in F. Without loss of generality, $u' \in H_2$. Suppose first that $v' \in H_2$. Then H_2 has a Hamiltonian path from u' to v'. This implies a perfect matching in $H_2 - \{u', v'\}$. Together with (u, u'),(v, v'), the remainder (all of the edges besides f) of the perfect matching in $H_1 - F_1'$, and perfect matchings in each H_i, $i \neq 1, 2$, we obtain a perfect matching in $A_{n,3} - F$, which is a contradiction. If, on the other hand, $v' \notin H_2$, we may assume that $v' \in H_3$. Then there exists a cross edge from $z \in H_2$ to $z' \in H_3$ such that $(z, z') \notin F$. Once again, there are Hamiltonian paths from u' to z in H_2 and from z' to v' in H_3. By adding f, (u, u'), (z, z'), and (v, v'), we create an even cycle whose only faulty edge is f. Thus we have a perfect matching on the vertices in the cycle. Adding the other edges in the perfect matching in $H_1 - F_1'$ and the perfect matchings in each H_i for $i \neq 1, 2, 3$, we obtain a perfect matching in $A_{n,3} - F$, which is a contradiction.

So F_1' forms a trivial conditional matching preclusion set in H_1, thus in H_1 there is a path u–w–v, with u and v adjacent such that the $4(n-3)-3$ faulty edges in H_1 are precisely the edges incident to either u or v not including (u, w) and (v, w). Each of u and v are incident to $n-3$ cross edges. But we only have $2(n-3)-1$ cross edges that are faulty, so we may assume that a cross edge (u, u') is not faulty. For notational convenience, assume that u' is in H_2. Now, by Corollary 1.6, $H_1 - \{u, w, v, f\}$ has a Hamiltonian cycle C. Note that C is an odd cycle. Clearly, as before, we can find a vertex y on C such that (y, y') is a cross edge that is not faulty and that y' is in H_2. Let P be a Hamiltonian path from u' to y' in H_2. Now it is easy to get a perfect matching for $A_{n,3}$: We use (w, v), (y, y'), (u, u'), a perfect matching induced by $C - \{y\}$, a perfect matching induced by $P - \{u', y'\}$, and a perfect matching from each H_i for $i = 3, 4, \ldots, n$. This finishes the proof. □

3. A boundary case: $A_{n,2}$

To complete the proof of the main result, we need to prove that it holds for the other boundary cases, that is, for $A_{n,2}$. One possibility is to use induction on n; however, there is no natural decomposition to aid the induction step. Certain steps in the proof require $n \neq 6$, so we treat the case $A_{6,2}$ separately, proving it in the Appendix. It is worthwhile to note that the proof for $A_{6,2}$ only affects $A_{n,k}$ with $n-k = 4$.

Theorem 3.1. *Let $n \geq 4$. Then $\mathrm{mp}_1(A_{n,2}) = 4n - 11$. Moreover, every optimal conditional matching preclusion set is trivial.*

Proof. For this boundary case, we decompose the graph into an array such that vertex ij occupies the entry in the intersection of the ith row and the jth column. Then the vertices of the graph are arranged in an $n \times n$ array, except for the main diagonal. The ith row and jth column, denoted by R_i and C_j, respectively, will consist of all of the vertices starting with i and ending in j, respectively. Then vertices in each R_i and in each C_j induce a graph isomorphic to K_{n-1}. Edges between vertices in the same column will be called *vertical*, edges between vertices in the same row will be *horizontal*.

Now assume that F is a set of edges with $|F| \leq 4n - 11$ such that $A_{n,2} - F$ does not have any isolated vertices. Edges in F are called *faulty*, other edges of $A_{n,2}$ are *good*. We will show that either $A_{n,2} - F$ has a perfect matching or F is a trivial conditional matching preclusion set. To complete the proof, we split into two cases based on the parity of n.

First assume that n is even. We will need the following result describing the matching preclusion number and the optimal conditional matching preclusion sets in the complete graph K_n:

Theorem 3.2 ([7]). *Let $n \geq 4$ be even. Then*

$$mp_1(K_n) = \begin{cases} \dfrac{n(n+2)}{8} & \text{if } n \in \{4, 6, 8\}, \\ 2n-5 & \text{if } n \geq 10. \end{cases}$$

Moreover, the optimal conditional matching preclusion sets are as follows: For $n \in \{4, 6, 8\}$ they can be edges of a complete subgraph $K_{n/2+1}$; for $n = 10$ they can be trivial or the edges of a complete subgraph K_6; for $n \geq 12$ they can only be trivial.

Note that Theorem 3.2 implies that $\mathrm{mp}_1(K_n) = 2n - 6$ if $n = 6$ or $n = 8$, and $\mathrm{mp}_1(K_n) = 2n - 5$ otherwise.

By Theorem 1.4, the case $n = 4$ is already done. We defer the case $n = 6$ to the Appendix, thus assume $n \geq 8$. Without loss of generality, at least half of the faults are vertical, therefore there are at most $2n - 6$ horizontal faults. Our goal, if F is not a trivial conditional matching preclusion set, is to select a set of vertical edges forming a matching in $A_{n,2} - F$ such that deleting their endpoints leaves an even number of vertices in each row with the property that each row has a perfect matching in the remaining vertices. Such a set of vertical edges will be called a *transversal*, and the existence of a transversal implies that $A_{n,2} - F$ has a perfect matching. So it is enough to show that a transversal fails to exist only when F is a trivial conditional matching preclusion set.

We think of the rows of $A_{n,2} - F$ as the vertices of a graph Q, with two rows adjacent if there is a good vertical edge between the rows so that each of these two rows has a perfect matching in the remaining vertices. Note that Q has an even number of vertices, and a perfect matching in Q implies the existence of a transversal and hence a perfect matching in $A_{n,2} - F$. However, in some cases we will need a transversal that does not correspond to a perfect matching in Q, i.e., there is a row containing the endpoints of several edges in the transversal.

We call a row R *constrained* if R has at least one vertex v such that $(R - v) - F$ does not have a perfect matching. In this case we call v a *constraining* vertex. Otherwise we say that v is *non-constraining*.

Suppose there are no constrained rows. There are $n - 2$ vertical edges between any two unconstrained rows, thus there is an edge between these rows in Q unless all these edges are faulty. Since there are at most $4n - 11$ faulty vertical edges, there can be at most three pairs of rows that do not have a good edge between them. Hence Q is a complete graph minus at most three edges on at least eight vertices, and therefore has a perfect matching.

Otherwise there must be at least one constrained row, call it R, and let v be a constraining vertex in R. Since $R - v$ is isomorphic to K_{n-2} and $\mathrm{mp}(K_{n-2}) = n - 3$ (see [3]), such a constrained row must have at least $n - 3$ faults. Therefore there are at most two constrained rows, and there are two such rows only if each has precisely $n - 3$ faults. Suppose that a row R has at least three non-constraining vertices. These three vertices are incident to $3(n - 2)$ vertical edges. But there are at most $(4n - 11) - (n - 3) = 3n - 8$ vertical faults. Therefore at least one vertical edge from R is good, so R is incident to an edge in Q. We now split into some subcases:

Case 1: Every vertex in $A_{n,2} - F$ has nonzero degree in its row.

Subcase 1a: There are exactly two constrained rows.

Without loss of generality let the constrained rows be R_1 and R_2. Each must contain exactly $n - 3$ faults. If there is a vertex v in R_1 such that $(R_1 - v) - F$ does not have a perfect matching, then it must have an isolated vertex (there are not enough faulty horizontal edges for any conditional matching preclusion set). In this case all of the faults are incident to a single vertex, and v is the only vertex not incident to any fault in R_1. Thus v is the only constraining vertex in R_1 (deleting any other vertex leaves at most $n-4$ faults). The same holds true in R_2. Therefore between the two constrained rows there are at least $n - 4$ vertical edges to choose from. There are $n - 3$ such edges between any constrained row and an unconstrained row and $n-2$ such edges between two unconstrained rows. Since there are exactly $2n-5$ vertical faults, there are at most two pairs of rows without good edges between them ($n - 4 + 2(n - 3) = 3n - 10 > 2n - 5$). Therefore Q is a complete graph with at least eight vertices minus at most two edges, so Q has a perfect matching.

Subcase 1b: There is only one constrained row.

Without loss of generality, R_1 is the only constrained row, and vertex $[1, 2]$ is of minimum degree in $R_1 - F$. Let $R' = R_1 - [1, 2]$.

Subcase 1b(i): $R' - F$ has no perfect matching.

We claim that in this case $R' - F$ must have an isolated vertex unless $n = 8$, which we discuss separately. Suppose to the contrary that $R' - F$ has no isolated vertices. Then Theorem 3.2 implies $\mathrm{mp}_1(K_{n-2}) = 2n - 9$ for $n \geq 12$ and $\mathrm{mp}_1(K_{n-2}) = 2n - 10$ for $n = 8$ and $n = 10$, so R' must have at least that many faults. Since there are at most $2n - 6$ horizontal faults, for $n \geq 12$ we get that at most three of them are incident to $[1, 2]$, which has minimum degree in $R_1 - F$, so the number of faulty edges in R_1 is at most $\lfloor \frac{3(n-1)}{2} \rfloor = \frac{3n-4}{2}$, which is less than $2n-6$ for $n > 8$. Hence in this case there are at most $2n - 7$ faults in R_1, so at most two of them are incident to $[1, 2]$, so the number of faulty edges in R_1 is at most $\frac{2(n-1)}{2} = n - 1 < 2n - 9$ for $n > 8$. Thus only the cases $n = 8$ and $n = 10$ remain. Consider $n = 10$. If there are fewer than $2n - 6 = 14$ faults in R_1 or there are at least $2n - 9 = 11$ faults in R', we get a contradiction in a similar way, so we may assume that there are exactly 10 faults in R' and 14 in R_1. Then by the characterization of optimal conditional matching preclusion sets in Theorem 3.2 we get that $R' - F$ is K_8 minus edges of a complete subgraph K_5, so there are five vertices of degree 3 in it, and the degree of $[1, 2]$ is 4 in $R_1 - F$, so the minimum degree in $R_1 - F$ is 3, contradicting the choice of $[1, 2]$. Finally consider $n = 8$. Using the same argument, we get a contradiction if there are fewer than $2n - 6 = 10$ faults in R_1 or there are at least $2n - 9 = 7$ faults in R'. Again by Theorem 3.2 we get that $R' - F$ is K_6 minus edges of a complete subgraph

K_4 (i.e., the complete bipartite graph $K_{4,2}$ plus an edge connecting the vertices on the side having two vertices), so there are four vertices of degree 2 in it, and the degree of [1, 2] is 2 in $R_1 - F$, so the minimum degree in $R_1 - F$ is 2. Now unless [1, 2] is also joined to the two vertices of degree 5 in $R' - F$, choosing one of the other vertices of minimum degree in $R_1 - F$ instead of [1, 2] results in $R' - F$ having a perfect matching, which we discuss below in Subcase 1b(ii). So the only remaining case is when $R_1 - F$ is isomorphic to $K_{5,2}$ plus an edge between the vertices on the smaller side. Then to find a perfect matching in $A_{n,2} - F$, we need to use vertical edges for at least three of the five vertices of degree 2 in $R_1 - F$ (after which we have a perfect matching on the remaining four vertices). Since there are exactly 11 vertical faults, it is easy to see that we can find good vertical edges to three of these vertices in R_1 that all go to different rows. Then in the remaining four rows there are six vertical edges between any two of them, so in Q there is at most one edge missing between them, so there is a matching covering these four vertices in Q. Using corresponding good vertical edges gives a transversal, and since no rows other than R_1 contain horizontal faults, we find a perfect matching in $A_{n,2} - F$.

Now assume without loss of generality that [1, 3] is isolated in $R' - F$. Then [1, 2] and [1, 3] have degree at most 1 in $R_1 - F$ and therefore have degree exactly 1. Moreover, edge ([1, 2], [1, 3]) is not faulty. In that case R_1 has at least $2(n-3)$ faults, all incident to [1, 2] or [1, 3], and there are no other horizontal faults. Therefore vertices $\{[1, 4], \ldots, [1, n]\}$ induce a complete graph in $R_1 - F$, so none of them are constraining. Thus R_1 has $n - 3$ vertices that can be used for a vertical edge. Since there are exactly $2n - 5$ vertical faults, there are at most two row pairs that do not have edges between them. Thus Q is at worst a complete graph minus two edges on at least eight vertices, so Q has a perfect matching.

Subcase 1b(ii): $R' - F$ does have a perfect matching.
In this case [1, 2] can be used for a vertical edge. Also, there is at least as many additional vertices that can be used for a vertical edge as the degree of [1, 2] in $R_1 - F$. This is because if edge ([1, 2], [1, x]) is not faulty, then for some vertex y, edge ([1, x], [1, y]) is in the perfect matching of $R' - F$. Then adding ([1, 2], [1, x]) and deleting ([1, x], [1, y]) from the perfect matching produces a perfect matching of $(R_1 - [1, y]) - F$. Thus [1, y] is also non-constraining. Therefore if d is the degree of [1, 2] in R', then at least $d + 1$ vertices in R_1 can be used for a vertical edge. We will show that either R_1 is incident to an edge in Q or there is a trivial conditional matching preclusion set induced by three vertices all in R_1.

Recall that if R_1 has at least three non-constraining vertices, then R_1 is incident to an edge in Q. This will happen if [1, 2] has degree at least 2 in $R_1 - F$. Otherwise [1, 2] has degree exactly 1 in $R_1 - F$. Without loss of generality, [1, 2] is adjacent only to [1, 3] and [1, 3] is matched to [1, 4] in a perfect matching of $R' - F$. Then [1, 2] and [1, 4] are both non-constraining and we have $n - 3$ horizontal faults all incident to [1, 2]. Furthermore, [1, 2] and [1, 4] are incident to $2(n - 2)$ vertical edges. If one of these edges is not faulty, then R_1 is incident to an edge in Q, thus assume that all of them are faulty. Then we have accounted for $3n - 7$ faulty edges, so there are at most $n - 4$ left. Let $A = R_1 - [1, 2] - [1, 3]$. If vertex [1, x] is non-constraining for any $x \in \{5, 6, \ldots, n\}$, then R_1 is incident to an edge in Q. Otherwise for each such x, either $(A - [1, x]) - F$ has an isolated vertex or $A - [1, x]$ has some conditional matching preclusion set (not necessarily trivial).

If $A - [1, x]$ has a conditional matching preclusion set for at least one such x, then it contains at least $2(n - 4) - 6$ faults for $n \geq 10$ ($A - [1, x]$ has $n - 4$ vertices). For $n > 10$ we have $2(n-4) - 6 > n - 4$, so we have a contradiction. When $n = 10$, we have $2(n - 4) - 6 = n - 4$, so the only possibility left is that $A - [1, x]$ is isomorphic to K_6 with edges of a K_4 subgraph removed, and we have identified all faults. Thus [1, x] is adjacent to every other vertex in A, and it is easy to see that every vertex of the K_4 is non-constraining. When $n = 8$, the graph $A - [1, x]$ is isomorphic to K_4, so the only conditional matching preclusion set in it is edges of a K_3. Thus without loss of generality assume that $(A - [1, 8]) - F$ is isomorphic to $K_{1,3}$. Note that we have found three more faulty edges, so there is at most one other fault in the graph. Hence [1, 8] is adjacent to at least two of the three leaves in $K_{1,3}$, and again it is easy to see that those three leaves are all non-constraining. Thus in each case R_1 has three non-constraining vertices, so it is incident to an edge in Q.

Thus assume that $(A - [1, x]) - F$ has an isolated vertex for every $x \in \{5, 6, \ldots, n\}$. Consider $(A - [1, 5]) - F$ with isolated vertex [1, y]. Then A contains $n - 5$ faults incident to [1, y] (with edge ([1, y], [1, 5]) not among them). Then we have accounted for all but one of the faults. First suppose $y = 4$. If edge ([1, 4], [1, 5]) is also faulty, then F is a trivial conditional matching preclusion set induced by the path [1, 2]–[1, 3]–[1, 4]. Thus assume that ([1, 4], [1, 5]) is a good edge. But then $B = A - [1, 4] - [1, 5]$ has at most one fault on $n - 5$ vertices. Since $n \geq 8$, the graph B is a complete graph on at least three vertices with at most one faulty edge. So $B - F$ must have an almost-perfect matching, and the vertex left out of the matching is non-constraining, so R_1 is still incident to an edge in Q. Therefore we may assume that $y \neq 4$. Without loss of generality, $y = 6$. Once again, there is at most one faulty edge unaccounted for. Consider $A - [1, 6]$, which must have an isolated vertex, say [1, z]. But this implies another $n - 5$ faulty edges (similarly to the above, edge ([1, z], [1, 6]) is not among them). Since there is only one more faulty edge possible, this means $n = 6$. But we have assumed $n > 6$, so this is a contradiction.

Thus, as claimed, either F is a trivial conditional matching preclusion set or R_1 is incident to an edge in Q. Therefore, assume that R_1 indeed is incident to an edge in Q.

Each of the other pairs of rows has $n - 2$ edges between them. Since there are at most $3n - 8$ vertical faults, there are at most two pairs without such an edge among the remaining $n - 2$ rows. Pick an edge incident to R_1, and then the induced subgraph of Q on the other $n - 2 \geq 6$ vertices is complete with at most two edges missing. So Q has a perfect matching, which completes Case 1.

Case 2: A vertex in $A_{n,2} - F$ has degree 0 in its row.

Without loss of generality, assume that vertex [1, 2] is isolated in R_1. Then [1, 2] is incident to a good edge, without loss of generality, to ([1, 2], [3, 2]). Since R_1 has at least $n - 2$ faults, there are at most $n - 4$ other horizontal faults. Thus R_1 is the only constrained row, and $(R_1 - [1, 2]) - F$ must have a perfect matching. Therefore we can use the good vertical edge $([1, 2], [3, 2])$ in the transversal. There are at most $(4n - 11) - (n - 2) = 3n - 9$ vertical faults, so as before, each of the other pairs of rows has $n - 2$ edges between them, so there are at most two pairs without a good vertical edge among the remaining $n - 2$ rows. Hence $Q - R_1 - R_3$ is a complete graph on $n - 2$ vertices minus at most two edges, so it has a perfect matching.

This completes the proof for even n.

Now assume that n is odd and $A_{n,2} - F$ has no perfect matching. We say that a line (row or column) is *bad* if it does not have a perfect matching. Similarly to the even case, we may assume, without loss of generality, that at most half of the faults, so at most $2n - 6$, are horizontal. A bad row has at least $n - 2$ faults, so there is at most one bad row. Moreover, if there are no bad rows, then $A_{n,2} - F$ has a perfect matching. Thus there is precisely one bad row; without loss of generality we may assume that R_1 is bad, vertex $[1, 2]$ has minimum degree in $R_1 - F$, and vertex $[1, 3]$ has minimum degree in $(R_1 - [1, 2]) - F$.

Let $L' = R_1 - [1, 2] - [1, 3]$. We claim that $L' - F$ has a perfect matching.

This can be easily checked for $n = 5$, since then R_1 is just K_4 containing at most four faulty edges, so assume $n \geq 7$. We claim that $L' - F$ has no isolated vertex. Otherwise this vertex has degree at most 1 in $(R_1 - [1, 2]) - F$ and at most 2 in $R_1 - F$, hence by the minimality of the degrees of $[1, 2]$ and $[1, 3]$, there are at least $3(n - 4)$ faulty edges in R_1. However, $3n - 12 > 2n - 6$ for $n > 6$, so this is not possible. Hence if $L' - F$ has no perfect matching, then the faulty edges in L' must form a conditional matching preclusion set, so there are at least $2n - 12$ faulty edges in L' ($2n - 11$ for $n = 7$ or $n \geq 13$). This leaves at most six faulty edges incident to $[1, 2]$ or $[1, 3]$, so there are at most three faulty edges between $[1, 3]$ and L'.

If there is exactly one faulty edge between $[1, 3]$ and L', then there is at most one faulty edge incident to every vertex in L', so there are at most $\frac{n-3}{2}$ faulty edges in L', but $\frac{n-3}{2} < 2n - 12$ for $n > 7$ and $\frac{n-3}{2} = 2 < 3 = 2n - 11$ for $n = 7$.

If there are exactly two faulty edges between $[1, 3]$ and L', then there are at most two faulty edges incident to every vertex in L', so there are at most $\frac{2(n-3)}{2} = n - 3$ faulty edges in L'. But $n - 3 < 2n - 12$ for $n > 9$, so only the cases $n = 7$ and $n = 9$ remain. For $n = 7$ the graph $L' - F$ must be $K_{1,3}$, hence $[1, 3]$ could not have minimum degree in $(R_1 - [1, 2]) - F$. For $n = 9$ we have $n - 3 = 2n - 12$, so the faulty edges in L' must form an optimal conditional matching preclusion set, so by Theorem 3.2 they are edges of a K_4 subgraph, so there are three faulty edges incident to these vertices in L', again contradicting the minimality of the degree of $[1, 3]$.

Finally, if there are exactly three faulty edges between $[1, 3]$ and L', then there must be exactly three faulty edges incident to $[1, 2]$, and there must be $2n - 12$ faulty edges in L' forming an optimal conditional matching preclusion set. This implies that $n = 9$ or $n = 11$, since for other values of n an optimal conditional matching preclusion set has at least $2n - 11$ edges. Again by Theorem 3.2 the faulty edges in L' must be edges of a $K_{(n-1)/2}$. For $n = 9$ this means that $L' - F$ has four vertices of degree 2, and since $[1, 3]$ has degree 3 in $(R_1 - [1, 2]) - F$, it could not have minimum degree. For $n = 11$ we get that $L' - F$ has five vertices of degree 3, so $[1, 3]$ with degree 5 could not have minimum degree in $(R_1 - [1, 2]) - F$. This finishes the proof that $L' - F$ has a perfect matching.

We continue by splitting into several cases.

Case 1: Every edge e in row R_i lies in a perfect matching of $(R_i - F) + e$ for every $i > 1$.

Subcase 1a: $[1, 2]$ is isolated in $R_1 - F$.

Then edge $([1, 2], [k, 2])$ is good for some k, since $[1, 2]$ is not isolated in $A_{n,2} - F$. Suppose that $([1, 3], [m, 3])$ is also good for some $m > 1$. If k and m can be chosen to be the same, then the following is a perfect matching in $A_{n,2} - F$: edges $([1, 2], [k, 2])$, $([1, 3], [k, 3])$, a perfect matching in $L' - F$, perfect matchings in the other rows, and the rest of a perfect matching in row R_k with the edge $([k, 2], [k, 3])$ specified. If k and m have to be different and there exists a good edge between R_k and R_m not in C_2 or C_3, say $([k, j], [m, j])$, then the following is a perfect matching in $A_{n,2} - F$: edges $([1, 2], [k, 2])$, $([k, j], [m, j])$, $([m, 3], [1, 3])$, a perfect matching in $L' - F$, and perfect matchings in the other rows with one edge specified in each of R_k and R_m. Otherwise the $n - 4$ edges of the form $([k, j], [m, j])$ for $j \notin \{2, 3, k, m\}$ are all faults.

This argument works for any pair of good vertical edges of $[1, 2]$ and $[1, 3]$. Thus suppose that $[1, 2]$ has degree $r > 0$ in $C_2 - F$ and $[1, 3]$ has degree s in $C_3 - F$. We claim that $rs \leq 1$. As above, if $[1, 2]$ and $[1, 3]$ have good vertical edges to the same row, then $A_{n,2} - F$ has a perfect matching. Otherwise we must have $r + s \leq n - 1$, so we have at least $2(n - 2) - r - s \geq n - 3$ vertical faults in C_2 and C_3. Hence the $n - 2$ horizontal faults in R_1 leave at most $2n - 6$ vertical faults in the other columns. For each pair of good vertical edges from $[1, 2]$ and $[1, 3]$, we either produce a perfect matching or identify $n - 4$ faults between the corresponding rows. Thus we can find $rs(n - 4)$ vertical faults outside C_2 and C_3. First assume $n \geq 7$. If $rs > 2$, then $3(n - 4) > 2n - 6$ gives a contradiction, and if $rs = 2$, then we have $2(n - 2) - 1 - 2 = 2n - 7$ vertical faults in C_2 and C_3, leaving at most $n - 2$ vertical faults in the other columns, but again $2(n - 4) > n - 2$. Hence we must have $rs \leq 1$. Next consider $n = 5$, so $r + s \leq 4$, $r, s \leq 3$. Since there are at most 9 faults in $A_{n,2}$, that leaves at most four faults in the columns outside C_2 and C_3. Note that we have at least two candidates for j whenever at least one of k or m is either 2 or 3 in the above argument, so only $\{k, m\} = \{4, 5\}$ yields only one vertical fault between the corresponding rows. Thus we get a contradiction similar to the above when $rs \geq 3$. When $rs = 2$, we have three vertical faults in C_2 and C_3, leaving at most three vertical faults in the other columns. We can choose a pair of good vertical edges of $[1, 2]$ and $[1, 3]$ in two ways, and at least one of them gives two choices for j for the good edge between R_k and R_m. Thus we either find a perfect matching in $A_{n,2} - F$, or identify all 9 faults. In the latter case we get that every fault in R_1 is incident to $[1, 2]$, so $[1, 4]$ can be chosen instead of $[1, 3]$ (both having minimum degree in $(R_1 - [1, 2]) - F = R_1 - [1, 2]$), and then repeating the above argument yields a perfect matching in $A_{n,2} - F$, contradiction. Thus again $rs \leq 1$.

Therefore [1, 3] has degree at most 1 in $C_3 - F$, and if its degree is 1, then [1, 2] also has degree 1 in $C_2 - F$. If [1, 3] is isolated in $C_3 - F$, then it cannot be isolated in $R_1 - F$. On the other hand, if [1, 3] has degree 1 in $C_3 - F$, then we identified $n - 3$ vertical faults in C_3, $n - 3$ more vertical faults in C_2, and $n - 4$ vertical faults between the two corresponding rows. Since there are $n - 2$ horizontal faults incident to [1, 2], this leaves at most one fault in $R_1 - [1, 2]$. In either case there is at least one good edge incident to [1, 3] in $R_1 - [1, 2]$, without loss of generality it is ([1, 3], [1, 4]).

Again without loss of generality, edge ([1, 4], [1, 5]) is in a perfect matching of $L' - F$. Adding edge ([1, 3], [1, 4]) and deleting edge ([1, 4], [1, 5]) from that matching gives a perfect matching in $(R_1 - \{[1, 2], [1, 5]\}) - F$. Repeating the argument above using [1, 5] instead of [1, 3] either gives a perfect matching in $A_{n,2} - F$ or shows that [1, 5] has degree at most 1 in $C_5 - F$ and identifies another $n - 3$ vertical faults in C_5. This process can be repeated for each good edge incident to [1, 3] or [1, 5] in $R_1 - [1, 2]$. We have found $n - 2$ faults in R_1 and $2(n - 3)$ faults in C_3 and C_5. If at least one of [1, 3] and [1, 5] is incident to a good edge in its column (which must be the only good edge incident to it in that column), then [1, 2] also has degree 1 in $C_2 - F$, giving $n - 3$ more vertical faults in C_2 and $n - 4$ vertical faults between the corresponding rows. This is $5n - 15 > 4n - 11$ faults in total, contradiction. Hence [1, 3] and [1, 5] must be both isolated in $C_3 - F$ and $C_5 - F$, respectively, and we found $3n - 6$ faults overall, with at most $n - 5$ faults remaining. Therefore [1, 3] has at least one more good edge incident to it in $R_1 - [1, 2]$, giving at least $n - 3$ vertical faults in that column, contradiction.

Subcase 1b: [1, 2] is not isolated in $R_1 - F$.

Since [1, 2] has minimum degree in $R_1 - F$, this means that the set of faulty edges in R_1, denote it by F_1, contains some conditional matching preclusion set. Thus the number of faulty edges in R_1 is at least $\mathrm{mp}_1(K_{n-1})$, which is $2n - 8$ for $n = 7$ or $n = 9$ and $2n - 7$ otherwise. Suppose that F_1 contains a trivial conditional matching preclusion set in R_1. Then without loss of generality, it is induced by the path [1, 2]–[1, 4]–[1, 3], with [1, 2] and [1, 3] having degree 1. If [1, 2] and [1, 3] are both isolated in $C_2 - F$ and $C_3 - F$, respectively, then F must be exactly this trivial conditional matching preclusion set, so assume that one of them, say [1, 2], is not isolated in its column.

Then we can repeat the argument in Subcase 1a with either $2n - 8$ or $2n - 7$ faults in R_1 rather than $n - 2$. For $n > 5$ we have $2n - 8 > n - 2$ and for $n = 5$ we have $2n - 7 = n - 2$, so we get a contradiction in a similar way (with a slight modification for $n = 5$).

Now we consider the case when F_1 does not contain a trivial conditional matching preclusion set, so $|F_1| \geq 2n - 7$. For $n = 5$ this is not possible (deleting at least three edges in K_4 leaves either a perfect matching, an isolated vertex, or $K_{1,3}$), so we have $n \geq 7$. If at least one of [1, 2] and [1, 3] is not isolated in its column, then we can repeat the argument in Subcase 1a and get a contradiction. If both of them are isolated in its respective column, then we have $2(n-2)$ faults in those columns, so there are no other faults in any other column or row. Note that edge ([1, 2], [1, 3]) cannot be good in R_1, since that gives a perfect matching in $R_1 - F$. Since [1, 2] and [1, 3] are not isolated in $R_1 - F_1$, and F_1 does not contain a trivial conditional matching preclusion set, the set of all neighbors of [1, 2] and [1, 3] in $R_1 - F_1$ cannot be just one vertex. We want to find different neighbors of [1, 2] and [1, 3] in $R_1 - F_1$ which are not matched to each other in a perfect matching of $L' - F$. This is easy to do if at least one of [1, 2] and [1, 3] has degree at least 3 in $R_1 - F$ (choose a neighbor of the other vertex first, then pick a neighbor not matched to it). If both degrees are at most 2, but they have at least three distinct neighbors in $R_1 - F$, then first pick a neighbor adjacent to just one of them, then pick a neighbor for the other of the remaining two not matched to it. If [1, 2] and [1, 3] have exactly two neighbors, say [1, 4] and [1, 5], which are matched to each other in all perfect matchings of $L' - F$, then there are two possibilities. If both [1, 2] and [1, 3] have degree 2, then replacing edge ([1, 4], [1, 5]) with edges ([1, 2], [1, 4]) and ([1, 3], [1, 5]) gives a perfect matching of $R_1 - F_1$, contradiction, while if [1, 2] has degree 1 and [1, 3] has degree 2, then there are $(n - 3) + (n - 5) = 2n - 8$ faults incident to [1, 2] and [1, 3], so there is at most one fault in L'. Since $n \geq 7$, there is a perfect matching in $L' - F - ([1, 4], [1, 5])$, so there is a perfect matching in $L' - F$ not matching [1, 4] to [1, 5]. Hence we can find two different vertices, say [1, 4] and [1, 5], such that edges ([1, 2], [1, 4]) and ([1, 3], [1, 5]) are both good in R_1, and, without loss of generality, let edges ([1, 4], [1, 6]) and ([1, 5], [1, 7]) be part of a perfect matching in $(R_1 - \{[1, 2], [1, 3]\}) - F$. Then replacing edges ([1, 4], [1, 6]) and ([1, 5], [1, 7]) with edges ([1, 2], [1, 4]) and ([1, 3], [1, 5]) gives a perfect matching in $(R_1 - \{[1, 6], [1, 7]\}) - F$, so we can repeat the argument of Subcase 1a with [1, 6] and [1, 7] playing the role of [1, 2] and [1, 3]. Since there are no vertical faults in C_6 and C_7, this immediately gives a perfect matching in $A_{n,2} - F$.

Case 2: Assume the premise of Case 1 is false, i.e., for some $i > 1$, there is an edge e in row R_i such that there exists no perfect matching in $(R_i - F) + e$ containing e (even if e is in F).

This means that $F - \{e\}$ is a matching preclusion set in the graph obtained from R_i by deleting the endpoints of e, which is isomorphic to K_{n-3}. Hence there are at least $n - 4$ horizontal faults in R_i. Since there at most $2n - 6$ horizontal faults and R_1 contains at least $n - 2$ horizontal faults as well, the number of horizontal faults must be exactly $n - 2$ in R_1 and $n - 4$ in R_i (hence e is not a fault), and there are no horizontal faults in the other rows. For $m \geq 3$ every optimal matching preclusion set in K_{2m} is trivial, while in K_4 an optimal matching preclusion set is either trivial or forms the edges of a triangle (leaving $K_{1,3}$). Hence vertex [1, 2] must be isolated in $R_1 - F$ for $n \geq 7$, and for $n = 5$ it is either isolated or has degree 1. In R_i the faulty edges must either be all incident to a single vertex having degree 2 or form a triangle (this can only happen when $n = 7$). In each case it is easy to check that in R_i at most two edges incident to $[i, 2]$ cause problems, so at most two values cannot be chosen for j.

Now we can repeat the argument of Subcase 1a while trying to avoid using $k = i$. Consider $n \geq 7$ first. Note that $R_1 - [1, 2]$ has no faults, so every vertex in $R_1 - [1, 2]$ can play the role of [1, 3] in the argument. First assume that k can be chosen to be different from i (i.e., ([1, 2], $[k, 2]$) is a good edge and $k \neq i$). If there is no column such that m can be chosen to be equal to k, then there must be $n-3$ vertical faulty edges between R_1 and R_k (all edges of the form $([1, t], [k, t])$ for $t \geq 3$, $t \neq k$), leaving $n-2$ other vertical faults. Thus if there are at least three such choices for k, then we find $3(n-3) > 2n-5$ vertical faults, contradiction. Similarly if there are exactly two such k, we also get $n-5$ vertical faults in C_2, again giving $2(n-3)+(n-5) = 3n-11 > 2n-5$ vertical faults. If there is only one such k, we get $n-4$ vertical faults in C_2 and $(n-3)+(n-4) = 2n-7$ vertical faults overall. Thus there are only two more vertical faults, so it is easy to find a perfect matching in $A_{n,2}-F$ using the rest of the argument. The only remaining case is when $[1, 2]$ is only joined to $[i, 2]$, giving $n-3$ vertical faults in C_2. Since in R_i only two values cannot be chosen for j, trying to find $k = m = i$ yields $n-5$ more vertical faults between R_1 and R_i, leaving only three more vertical faults. Then it is easy to finish the argument and find a perfect matching in $A_{n,2} - F$. The case $n = 5$ can be done similarly, we omit the (tedious) details.

Therefore the case where n is odd is complete and the proof for $k = 2$ is finished. □

4. Conclusion

In this paper, we solved the conditional matching preclusion problem for $A_{n,k}$ as well as classified all the optimal conditional matching preclusion sets. The proof utilized the known result for $A_{n,n-2}$ to cover one boundary and used known results on fault Hamiltonicity extensively through out the proof.

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Appendix. Proof of the Exceptional Case: $n = 6$, $k = 2$

In this appendix we tie up the proof of the base case (Theorem 3.1) and the main theorem (Theorem 2.1) and prove the following theorem:

Theorem A.1. $mp_1(A_{6,2}) = 13$. *In addition, every optimal conditional matching preclusion set is trivial.*

Proof. We proceed as in the proof of the even case of Theorem 3.1, making adjustments when necessary. As before, if there are no constrained rows, then the auxiliary graph Q will have a perfect matching. Thus, there must be at least one constrained row (note that there are still at most two constrained rows). As in Theorem 3.1, we split into cases.

CASE 1: Every vertex in $A_{6,2} - F$ has nonzero degree in its row.

Subcase 1a: There are exactly two constrained rows.

Without loss of generality, R_1 and R_2 are the two constrained rows. Then each contains exactly three faults. If there is a vertex v in R_1 such that $(R_1 - v) - F$ does not have a perfect matching, then it either has an isolated vertex or is a $K_{1,3}$, as there are at most three faults. In either case, we have at most two constraining vertices in R_1, and the same holds for R_2. Therefore there are at least two vertical edges to choose from between a constrained row and an unconstrained row, and there are four such edges between any two unconstrained rows (there might not be any between the two constrained rows). Since there are exactly seven vertical faults, there are at most four pairs of rows without edges between them in Q $(1 \cdot 0 + 4 \cdot 2 = 8 > 7)$. Therefore Q is a complete graph on six vertices minus at most four edges, hence Q has a perfect matching.

Subcase 1b: There is only one constrained row.

Without loss of generality, R_1 is the only constrained row, and vertex $[1, 2]$ is of minimum degree in $R_1 - F$. Let $R' = R_1 - [1, 2]$.

Subcase 1b(i): $R' - F$ has no perfect matching.

Either $R' - F$ has an isolated vertex or $R' - F$ has some conditional matching preclusion set. If the former, then we can continue as in Theorem 3.1 and Q will have a perfect matching (a complete graph on six vertices minus at most two edges). Thus assume that $R' - F$ has a conditional matching preclusion set. Specifically, $R' - F$ is isomorphic to $K_{1,3}$ (K_4 has no non-optimal conditional matching preclusion sets). The three pendant vertices have degree at most 2 in $R_1 - F$, so the same is true for $[1, 2]$. But then $[1, 2]$ cannot be adjacent to all three of the pendant vertices. Thus one of the pendant vertices, and hence $[1, 2]$ has degree exactly 1 in $R_1 - F$. Therefore there are two cases. First, if $[1, 2]$ is adjacent to one of the pendant vertices, then if v is one of the other two pendant vertices, v also has minimum degree and $(R_1 - v) - F$ has a perfect matching, so we have reduced this case to Subcase 1b(ii), which is discussed below. In the other case, $[1, 2]$ is a fourth pendant vertex adjacent only to the central vertex.

Without loss of generality, assume that the central vertex in $R_1 - F$ is $[1, 4]$. Thus we have accounted for all six horizontal faulty edges with seven vertical faulty edges remaining. Suppose that at least three of the pendant vertices in $R_1 - F$ have good vertical edges to the same row, say row i. Then these edges, the edge between $[1, 4]$ and the remaining vertex, the edge between the other two vertices in row i, and a perfect matching involving the other four rows (there are four edges between any two of these rows, all unconstrained, so at most one pair does not have an edge in Q) gives a perfect matching in $A_{6,2} - F$.

Thus we may assume that no such row $i \neq 1$ has more than two good vertical edges to row R_1 in columns C_2, C_3, C_5, and C_6. This accounts for at least one vertical fault between row R_1 and rows R_2, R_3, R_5, and R_6 each. In addition, it forces at least two vertical faults between row R_1 and row R_4. Thus we have accounted for six vertical faults. There are no remaining horizontal faults and at most one remaining vertical fault.

Now, there must be some row that has two (rather than one or zero) good vertical edges (excluding the edge to $[1, 4]$) from row R_1, or else this requires another five vertical faults (one for each row). Without loss of generality, $([1, 2], [i, 2])$, $([1, 3], [i, 3])$ are good edges for some $i \neq 1$. Then one of the eight vertical edges from vertices $[1, 5]$ and $[1, 6]$ must be good (there are at most seven vertical faults). Then assume, without loss of generality, that $([1, 5], [j, 5])$ is a good edge with $j \neq i, 1$ (if $j = i$, we are done by the argument above). We can then construct a perfect matching in $A_{6,2} - F$ as follows: $([1, 4], [1, 6])$, $([1, 2], [i, 2])$, $([1, 3], [i, 3])$, $([1, 5], [j, 5])$, a perfect matching in $R_j - [j, 5]$, a vertical edge from row i to one of the other three rows, say row k (there is at most one vertical fault not involving row R_1 and i has at least two possible such edges with every such row), the edge between the remaining two vertices in R_i, a perfect matching in the rest of R_k, and an appropriate perfect matching involving the last two rows (both are unconstrained and have $4 > 1$ vertical edges between them). This completes Subcase 1b(i).

Subcase 1b(ii): $R' - F$ does have a perfect matching.

In this case $[1, 2]$ can be used for a vertical edge. We can repeat as in the proof of Theorem 3.1. The first possible difference to consider is if $A - [1, x]$ has a conditional matching preclusion set for $x \in \{5, 6\}$. But $A - [1, x]$ has only two vertices, so no conditional matching preclusion set is possible.

Thus assume $A - [1, x]$ has an isolated vertex for $x = 5$ and $x = 6$. Then $([1, 4], [1, 5])$ and $([1, 4], [1, 6])$ are faults and $\{[1, 2], [1, 4], [1, 3]\}$ gives a trivial conditional matching preclusion set in $R_1 - F$ with $[1, 2]$ and $[1, 4]$ the pendant vertices. As in Theorem 3.1, either $[1, 2]$ and $[1, 4]$ are isolated in their columns, giving a trivial conditional matching preclusion set for $A_{6,2}$, or R_1 is incident to an edge in Q. From here, this case is the same as in Theorem 3.1.

Case 2: A vertex in $A_{6,2} - F$ is isolated in its row.

This case is identical to the corresponding case in the proof of Theorem 3.1, thus completing the proof. □

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